Database on Tone Rows and Tropes
A short user's guide

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Abstract

Based on the theory developed and described in "Tone rows and tropes" we developed a database containing complete information on the 836 017 $D_{12} \times D_{12}$-orbits of tone rows. It can be accessed via `http://www.uni-graz.at/~fripert/db/`. We give some short hints on what can be done with this database and how to use it.

The *Database on tone rows and tropes* is publicly available[1] via the address

`http://www.uni-graz.at/~fripert/db/`.

It consists of the following components:

- Several `Perl`-programs can be used for different search routines.
- Different algorithms concerning group actions are implemented in one executable program written in "`SYMMETRICA`".[2]
- For some graphics the JavaScript VectorGraphics library "`wz_jsgraphics.js`" by Walter Zorn[3] is used. In order to print the graphics-output printing of the background must be enabled in the browser.
- Using pop-up windows further information can be derived, whence pop-up windows should also be enabled.
- The JavaScript library "`SoundManager 2`"[4] is used to obtain a simple acoustic representation of the tone rows.

1 Tone rows

The main objects in this database are the $D_{12} \times D_{12}$-orbits of tone rows. A tone row is a bijective mapping from $\{1, \ldots, 12\}$ into the set $\mathbb{Z}_{12}$ of twelve pitch classes. The orbit $(D_{12} \times D_{12})(f)$ of the tone row f can be described

- by the complete list of all elements $(\phi, \pi)f = \phi \circ f \circ \pi^{-1}$ for $(\phi, \pi) \in D_{12} \times D_{12}$, i.e. by all tone rows which can be obtained from f by applying any combination of transposing, inversion, cyclic shift and retrograde to f,
- by the normal form of f which is the lexicographically smallest element in the orbit of f,

[1] accessed July 5, 2014
[2] `http://www.algorithm.uni-bayreuth.de/en/research/SYMMETRICA/` accessed August 19, 2014.
[3] `http://www.walterzorn.de/en/jsgraphics/jsgraphics_e.htm` acessed August 19, 2014
[4] `http://www.schillmania.com/projects/soundmanager2/` accessed August 19, 2014

- by its number in the complete listing of all $D_{12} \times D_{12}$-orbits of tone rows, which belongs to the set $\{1, 2, \ldots, 836\,017\}$.

For more details see Sections 3 and 3.1 of [Fripertinger and Lackner(2015)].

There are three different alphabets representing the set of pitch classes, therefore, it is possible to handle tone rows in three different numerical representations. As alphabet we use either $A_1 = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ or $A_2 = \{0,1,2,3,4,5,6,7,8,9,10,11\}$ or $A_3 = \{0,1,2,3,4,5,6,7,8,9,A,B\}$. A tone row is a vector or array of length 12 over A_i, $1 \leq i \leq 3$, so that each element of the alphabet is listed exactly once. Using A_1 or A_2 the numbers in this vector must be separated by a comma, whereas no commas are used when writing a tone row as a vector over A_3. E. g., the chromatic scale could be expressed as `1,2,3,4,5,6,7,8,9,10,11,12` over A_1, as `0,1,2,3,4,5,6,7,8,9,10,11` over A_2, and as `0123456789AB` over A_3. By using radio buttons, the alphabet must be chosen in order to fit to the input tone row. The user must take care to consider the right alphabet, to separate or not to separate with commas and to input a list of exactly twelve different pitch classes.

In some places there are additional display options which allow to write or display a tone row with German, English or Italian note names or in musical notation. Using the "Play"-button you can listen to this tone row, the "Stop"-button interrupts the playback.

1.1 $D_{12} \times D_{12}$-orbit of a tone row

In order to compute all elements of the $D_{12} \times D_{12}$-orbit of the tone row f it must be input either in numeric form or by choosing note names.

For each tone row h in the orbit of f an element (ϕ, π) of the acting group $D_{12} \times D_{12}$ is presented so that $h = (\phi, \pi)f = \phi \circ f \circ \pi^{-1}$. If you press the button "S" the stabilizer of the corresponding tone row in $D_{12} \times D_{12}$ is computed. The tone rows are ordered by the lexicographic order. There are at most $576 = 24^2$ tone rows in one orbit.

For more details see Sections 3.1, 3.2 and Remark 3.1 of [Fripertinger and Lackner(2015)].

1.2 $D_{12} \times D_{12}$-normal form of a tone row

The normal form h of the tone row f is the lexicographic smallest element in the orbit of f. Thus it is the tone row which occurs in first position in the listing of all elements of the $D_{12} \times D_{12}$-orbit. Knowing h we can search the database for all information on the orbit of f. For more details see Section 3.2 and Remark 3.2 of [Fripertinger and Lackner(2015)].

In order to compute the normal form of f it must be input either in numeric form or by choosing note names. Not only the normal form h is computed, but also an element (ϕ, π) of the acting group $D_{12} \times D_{12}$ so that $h = (\phi, \pi)f = \phi \circ f \circ \pi^{-1}$ or, equivalently, $f = (\phi^{-1}, \pi^{-1})h = \phi^{-1} \circ h \circ \pi$. The position of the input tone row f in the orbit of h with respect to the lexicographic order is indicated. Moreover we present the orbit of h in the chromatic circular representation and indicate the particular position of f in this polygon or, if necessary, in its mirror which is the polygon of $I \circ h$. Since f is an individual tone row we indicate its first note (or pitch class) by a circle around the vertex corresponding to this note (or pitch class), and the connection between the last note and the first note of f is not drawn in the polygon. Moreover we indicate how the pitch classes are labeled. Finally, we draw the trope structure of f according to this labeling of the pitch classes.

There are further buttons which allow to search the database for this tone row, to obtain all information about this tone row, or to show the orbit of this tone row.

1.3 All information about a tone row

It is possible to obtain all information about a particular tone row f which can be input either in numeric form or by choosing its number in the list of the $836\,017$ $D_{12} \times D_{12}$-orbits of tone rows. For the second method the first input field must be empty! The user will obtain the normal form of f, the orbit of f, all data like stabilizer type, trope structure, interval structure, diameter distance structure, chord diagram etc. in connection with this orbit, the tone row in musical notation, the row matrix of f, simultaneous information on bigger orbits of the form $(\mathrm{Aff}_1(\mathbb{Z}_{12}) \times D_{12})(f)$, $(D_{12} \times \mathrm{Aff}_1(\mathbb{Z}_{12}))(f)$, $(\mathrm{Aff}_1(\mathbb{Z}_{12}) \times \mathrm{Aff}_1(\mathbb{Z}_{12}))(f)$, $\mathfrak{A}_{12}(f)$ and $\mathfrak{D}_{12}(f)$. (For more details see sections 1.3, 2.3, 3.1, and 4.1 of [Fripertinger and Lackner(2015)].) Finally musical information on the orbits $(D_{12} \times D_{12})(f)$ and $\mathfrak{A}_{12}(f)$ is provided. The second one, is the biggest orbit containing f which is studied in our database. It collects the information on tone rows of up to eight different $D_{12} \times D_{12}$-orbits.

1.4 Search the database of $D_{12} \times D_{12}$-orbits of tone rows

This is the main part of the database. It is possible to search for or to retrieve information on

- tone rows with given normal form,
- tone rows with given interval structure,
- tone rows with prescribed number of different intervals in their interval structure,
- tone rows with given trope structure,
- tone rows with trope numbers from a prescribed subset of $\{1, \ldots, 35\}$,
- tone rows with prescribed number of different trope numbers in their trope structure,
- tone rows with prescribed diameter distance,
- tone rows with prescribed number of different distances in their diameter distance,
- tone rows with prescribed stabilizer type,
- tone rows with given number in the list of all 836 017 orbits,
- all-interval rows,
- orbits of tone rows invariant under the quart-circle,
- orbits of tone rows invariant under the 5-step,
- orbits of tone rows invariant under the parameter exchange,
- orbits of derived tone rows,
- tone rows with given chord diagram.

If different search criteria are entered it is important to choose whether they should be connected with `AND` or `OR`.

By using radio-buttons the alphabet is chosen for representing the tone rows. There are some additional display options: If "Graphical" is chosen, then for each $D_{12} \times D_{12}$-orbit of tone rows the chromatic circular representation is shown. In this situation at most 100 search results are displayed on a single page. If "Normal" is chosen, then the chromatic circular representation is not shown, but up to 6000 search results can be displayed. If "Simple" is chosen, then the information on a single tone row is contained in a single line. This is the fastest display option. Again it is possible to display at most 6000 search entries. Each line in the output is separated by `|` in 14 columns. The first column contains the number of the orbit from $\{1, \ldots, 836\,017\}$, the second column the normal form of the tone row represented over the alphabet A_3, the third column the trope structure, the fourth column the number of different trope numbers, the fifth column the stabilizer type given by a number from $\{1, \ldots, 17\}$, the sixth column the interval structure represented over the alphabet A_3, the seventh column the number of different intervals, the eighth column the diameter distance structure, the ninth column the number of different diameter distances, the next three columns indicate whether the orbit is invariant under the quart circle, the five step, or the parameter exchange (where `0` means "no" and `1` means "yes" in the corresponding column). Finally, the last but one column indicates whether the row is a derived row or not. If it is derived, then it indicates the lengths of the subsequences from which it can be derived, otherwise it shows 0. The last column contains the number of the chord diagram from the set $\{1,\ldots,554\}$.

Using the "Normal" or "Graphical" display option for each search result it will be possible to open pop-up windows showing

- the chord diagram,
- the decomposition of the orbit into $D_{12} \times \langle R \rangle$-orbits,
- musical information on the $D_{12} \times D_{12}$-orbit and on the $\mathfrak{A}_{12}$-orbit containing this orbit,
- all elements of the orbit,
- the stabilizer of the normal form of the orbit,
- the trope structure of the normal form of the orbit,
- the chromatic circular representation of the orbit.

If there are more search results than could be displayed (more than 100 or 6000 respectively), then it is possible to continue the search pressing a button at the end of the page with the search results.

It is also possible to pass all the search results to other parts of the database:

- Get further information on bigger orbits
- Add further search criteria
- Search for musical information
- Simultaneous information on bigger orbits
- Chord diagrams and Gauss numbers
- Decomposition into $D_{12} \times \langle R \rangle$-orbits

Now we explain the more advanced input options:

1.4.1 The normal form

In order to input the normal form, first the alphabet must be chosen. According to this choice

1. either a complete normal form of a tone row must be input in numeric version (where each symbol of the chosen alphabet must appear exactly once).

   If a tone row is input which is not the normal form of its orbit it will not be found in the database. E.g. the sequence `2,3,4,5,6,7,8,9,10,11,12,1` also describes the chromatic scale, but is not contained in the database.

2. or a part (a subsequence) of the normal form of a tone row must be input, e.g. `10,11` or `1,3,4` (for the alphabet A_1) or `9,10` or `0,2,3` (for the alphabet A_2) or `9A` or `023` (for the alphabet A_3),

3. or a blank-separated list of parts of tone rows must be input, e.g. `4,5 2,7` (for the alphabet A_1) or `3,4 1,6` (for the alphabet A_2) or `34 16` (for the alphabet A_3). Using `AND` demands that all the different parts must simultaneously occur in one tone row, whereas using `OR` at least one of these parts must occur in one tone row.

The software will find a tone row if and only if the input pattern occurs in a normal form of a tone row.

1.4.2 The interval structure

In order to input the interval structure of a tone row

1. either a comma-separated list of 11 intervals from the set $\{1,2,\ldots,11\}$ must be input, if the alphabets A_1 or A_2 are chosen, or a sequence (without commas) of 11 intervals from the set $\{1,2,\ldots,9,A,B\}$ if the alphabet A_3 is considered. E.g. `1,1,2,7,9,7,2,2,11,4,11,3` or `11279722B4B3`
2. or a part of the interval-structure of a tone row must be input, e.g. `1,1,2,7` or `1127` which is an interval-structure of length 4,
3. or a blank-separated list of parts of an interval-structures must be input. E.g. `1,2,7 7,2,2 9,7,2` or `127 722 972` which consists in this case of three structures all of length three. Using `AND` demands that all the different parts must simultaneously occur in the interval structure of a tone row, whereas using `OR` at least one of these parts must occur in in the interval structure of a tone row.
4. It is also possible to replace the occurrence of an arbitrary interval by a dot `.`, of arbitrary many intervals by `.*` which includes also the situation of no intervals, of at least one interval by `.+`, or of at most one interval by `.?`. E.g., in order to search for an interval structure where there is exactly one digit between two intervals 6 enter `6,.,6` or `6.6`.

Searching for an interval-structure finds

- the occurrence of the input structure in the interval-structure of a tone row in normal form,
- but also in its retrograde,
- or in its inversion,
- or in its cyclic shifts. E.g. the interval-structure `11,3,1,1` is found in a tone row with interval structure `1,1,2,7,9,7,2,2,11,4,11,3` even though the four intervals occur at the end and the beginning of this sequence.

1.4.3 The trope structure

In order to prescribe the trope structure,

1. either a comma-separated list of 6 trope numbers from the set$\{1,2,\ldots,35\}$ must be input describing the complete trope structure of a tone row, e.g. `1,2,7,13,11,2`
2. or a part of the trope structure of a tone row must be input, e.g. `2,7` which is a trope structure of length 2.
3. or a blank-separated list of trope structures as in 2. must be input. E.g. `1,2 7 13` which consists of three structures of length 2, 1 and 1. Using `AND` demands that all the different parts must simultaneously occur in the trope structure of a tone row, whereas using `OR` at least one of these parts must occur in the trope structure of a tone row.
4. It is also possible to replace the occurrence of a single (numerical) digit by a dot `.` or by `\d`, of exactly two (numerical) digits by two dots `..` or by `\d\d`, of arbitrary many digits by `.*?` or by `\d*?`.

Searching for a trope structure finds

- the occurrence of the input structure in the trope structure of a tone row in normal form,
- but also in its retrograde,
- or in its cyclic shifts. E.g. the sequence `1,2,11` is found in a tone row with trope structure `1,2,7,13,11,2` even though the three trope numbers occur at the end and the beginning of this sequence in retrograde order.)

1.4.4 Trope numbers from a given set

A comma-separated list of trope numbers from 1 to 35 must be input. Only those tone rows are found the trope structure of which contains only numbers from this list.

1.4.5 The diameter distance structure

Let $f\colon \{1,\ldots,12\} \to \mathbb{Z}_{12}$ be a tone row, then its diameter distances are the distances of $f(7)$ and $f(1)$, of $f(8)$ and $f(2), \ldots,$ and of $f(12)$ and $f(6)$.

Let $a, b \in \mathbb{Z}_{12}$ be two distinct elements, then the distance of a and b is 1 if and only if the interval $b - a$ is 1 or 11. The distance is 2 if the interval $b - a$ is 2 or 10. The distance is 3 if the interval $b - a$ is 3 or 9. The distance is 4 if the interval $b - a$ is 4 or 8. The distance is 5 if the interval $b - a$ is 5 or 7. The distance is 6 if the interval $b - a$ is 6.

Let v_i be the distance of $f(i)$ and $f(i+6)$, $1 \leq i \leq 6$. They all belong to the set $\{1,\ldots,6\}$. The diameter distance structure is the D_6-orbit of the vector $(v_1,\ldots,v_6)$ where the dihedral group acts on the set of indices in the natural way.

In order to input the diameter distance structure

1. either a comma-separated list of 6 diameter distances describing the complete diameter distance structure of a tone row must be input, e.g. `1,2,4,4,5,4`,
2. or a part of the diameter distance structure of a tone row must be input, e.g. `2,4`, which is a diameter distance structure of length 2,
3. or a blank-separated list of distance structures as in 2. must be input. E.g. `1,2 4,5,4`, which consists of two structures of length 2 and 3. Using `AND` demands that all the different parts must simultaneously occur in the diameter distance structure of a tone row, whereas using `OR` at least one of these parts must occur in in the diameter distance structure of a tone row.
4. It is also possible to replace the occurrence of a single (numerical) digit by a dot `.` or by `\d`, of exactly two (numerical) digits by two dots `..` or by `\d\d`, of arbitrary many digits by `.*?` or by `\d*?`.

Searching for a diameter distance structure finds

- the occurrence of the input structure in the diameter distance structure of a tone row in normal form,
- but also in its retrograde,
- or in its cyclic shifts. E.g. the sequence `4,1,2` is found in a tone row with diameter distance structure `1,2,4,4,5,4` even though the three numbers occur at the end and the beginning of this sequence in retrograde order.)

1.4.6 Stabilizer type

Let f be a tone row, then its stabilizer U is the set of all $(\phi,\pi) \in D_{12} \times D_{12}$ so that $(\phi,\pi)f = f$. It is a subgroup of $D_{12} \times D_{12}$. The stabilizer of any element of the $D_{12} \times D_{12}$-orbit of f belongs to the conjugacy class of U. Hence, the conjugacy class of U is called the stabilizer type of this orbit.

There are 17 different conjugacy classes of subgroups of $D_{12} \times D_{12}$ which occur as stabilizers of tone rows. For each class $\tilde{U}_i$ we present the generator of one representative of this class, the order of this representative, the size of the class, i.e. the number of different groups in this conjugacy class, and the number of $D_{12} \times D_{12}$-orbits of tone rows with stabilizer type $\tilde{U}_i$.

In order to search for tone rows with given stabilizer type, determine the conjugacy class of the stabilizer by choosing a number from $\{1,\ldots,17\}$. The different stabilizer types are described in [Fripertinger and Lackner(2015)] or in the online documentation.

1.4.7 Derived tone rows

Let d be a number from $\{2, 3, 4, 6\}$. These are the nontrivial divisors of 12. Partition a tone row $f\colon \{1, \ldots, 12\} \to \mathbb{Z}_{12}$ into $q = 12/d$ segments of the form

$$(f(1), \ldots, f(d)), (f(d+1), \ldots, f(2d)), \ldots, (f((q-1)d+1), \ldots, f(12)).$$

The tone row f can be constructed from its first segment, if all the q segments of f can be constructed from the first segment by applying suitable combinations of transposition, inversion and retrograde (of the segment).

The tone row f is called a *derived row* if it can be partitioned into $12/d$ segments, where $d \in \{2, 3, 4, 6\}$, so that f can be constructed from its first segment.

If f is a derived row, then all elements in the orbit $(D_{12} \times \langle R \rangle)(f)$ are derived rows. We want to generalize this notion for $D_{12} \times D_{12}$-orbits.

The $D_{12} \times D_{12}$-orbit of f is called *derived*, if there exist representatives which are derived rows. Thus, the segment from which the row f can be constructed need not be the segment $(f(1), \ldots, f(d))$. It can be any segment of the form $(f(1+j), \ldots, f(d+j))$ for $0 \leq j < d$.

In order to search for derived rows decide which values of d are interesting. It is possible to choose any combination of the four possible values of d and to indicate by choosing `AND/OR/EXACT` whether the tone rows must be derived from each of these d's, at least one of these d's, or exactly from all these d's and no other d's.

1.5 Chord diagrams and Gauss words

Let $f\colon \{1, \ldots, 12\} \to \mathbb{Z}_{12}$ be a tone row. The preimages of the six tritone pairs $\{0, 6\}, \{1, 7\}, \ldots, \{5, 11\}$ partition the domain $\{1, \ldots, 12\}$ into six 2-sets. There are 554 D_{12}-orbits of these partitions which correspond to Gauss words. Gauss words can be represented as functions of restricted growth from $\{1, \ldots, 12\}$ to $\{1, \ldots, 6\}$ where each element of the range occurs exactly twice. Franck Jedrzejewski computed the serial groups of tone rows. All tone rows of the orbit $(D_{12} \times D_{12})(f)$ determine the same Gauss word and the same serial group as f. Hence, these are properties of the orbit of f. According to F. Jedrzejewski there are 26 non-isomorphic serial groups and their orders lie between 24 and 12!.

Using the database it is possible to

- to determine the chord diagrams of all tone rows given by the number of their ($D_{12} \times D_{12}$-orbit,
- display all $D_{12} \times D_{12}$-orbits of tone rows which yield a given chord diagram,
- display all $D_{12} \times D_{12}$-orbits of tone rows with given serial group. The serial groups are identified by their orders. There are three non-isomorphic serial groups of order 384, which are indicated as 384*a*, 384*b*, and 384*c*, and three non-isomorphic groups of order 768, indicated as 768*a*, 768*b*, and 768*c*.
- The different input blocks are connected with `AND`.

1.6 Multiplicities of intervals

The multiplicity of an interval indicates how often this interval occurs in the interval structure. For each interval i from 1 to 11 its multiplicity $a_i \in \{0, \ldots, 12\}$ can be prescribed. If

$$\sum_{i=1}^{11} a_i = 12,$$

then the program determines all tone rows the interval structures of which have multiplicities a_i for each $i \in \{1, \ldots, 12\}$. If

$$\sum_{i=1}^{11} a_i > 12,$$

then there are no tone rows with these multiplicities. If

$$\sum_{i=1}^{11} a_i < 12,$$

then the program determines all tone rows the interval structures of which have multiplicities at least a_i for each $i \in \{1, \ldots, 12\}$.

The interval structures of all-distances-twice rows are collected in a table.

2 The trope structure

Consider a tone row $f \colon \{1, \ldots, 12\} \to \mathbb{Z}_{12}$, then the tropes of f are the six "pairs" of hexachords

$$\begin{aligned}
\tau_1 &= \{\{f(1), f(2), f(3), f(4), f(5), f(6)\}, \{f(7), f(8), f(9), f(10), f(11), f(12)\}\},\\
\tau_2 &= \{\{f(2), f(3), f(4), f(5), f(6), f(7)\}, \{f(8), f(9), f(10), f(11), f(12), f(1)\}\},\\
\tau_3 &= \{\{f(3), f(4), f(5), f(6), f(7), f(8)\}, \{f(9), f(10), f(11), f(12), f(1), f(2)\}\},\\
\tau_4 &= \{\{f(4), f(5), f(6), f(7), f(8), f(9)\}, \{f(10), f(11), f(12), f(1), f(2), f(3)\}\},\\
\tau_5 &= \{\{f(5), f(6), f(7), f(8), f(9), f(10)\}, \{f(11), f(12), f(1), f(2), f(3), f(4)\}\},\\
\tau_6 &= \{\{f(6), f(7), f(8), f(9), f(10), f(11)\}, \{f(12), f(1), f(2), f(3), f(4), f(5)\}\}.
\end{aligned}$$

Therefore, a tone row f induces the *trope sequence* $t_f \colon \{1, \ldots, 6\} \to \mathcal{T}$, $t_f(i) = \tau_i$, $1 \leq i \leq 6$, where $\mathcal{T}$ is the set of all tropes in $\mathbb{Z}_{12}$. If we replace in the trope sequence of f the tropes by the numbers of their D_{12}-orbits, we obtain a function $s_f \colon \{1, \ldots, 6\} \to \{1, \ldots, 35\}$, the *trope number sequence* of f, where $s_f(i)$ is the number of the orbit $D_{12}(\tau_{i-1})$, $1 \leq i \leq 6$.

The trope structure of the $D_{12} \times D_{12}$-orbit of f is the D_6-orbit of s_f under the natural action of the dihedral group on the domain of s_f. For more details see Section 3.5 of [Fripertinger and Lackner(2015)].

In order to construct tone rows from their trope structure or to explore the set of all trope structures it is possible

- to find all tone rows with prescribed multiplicities of trope numbers in their trope structure,
- or to determine all tone rows the trope structures of which can be constructed from six given sets.

2.1 Multiplicities of trope numbers

The multiplicity of a trope number indicates how often this trope number occurs in the trope structure. For each trope number i from 1 to 35 its multiplicity $a_i \in \{0, \ldots, 6\}$ can be prescribed. If

$$\sum_{i=1}^{35} a_i = 6,$$

then the program determines all tone rows the trope structures of which have multiplicities a_i for each $i \in \{1, \ldots, 35\}$. If

$$\sum_{i=1}^{35} a_i > 6,$$

then there are no tone rows with these multiplicities. If

$$\sum_{i=1}^{35} a_i < 6,$$

then the program determines all tone rows the trope structures of which have multiplicities at least a_i for each $i \in \{1, \ldots, 35\}$.

2.2 Construction of trope structures

After inputting X_i, the set of trope numbers which may occur in i-th position of a trope structure for $i \in \{1, \ldots, 6\}$, all orbits of tone rows are constructed the trope structures of which satisfy these constraints. The sets X_i are written as comma-separated lists of numbers from $\{1, \ldots, 35\}$.

For each sequence $(t_1, \ldots, t_6) \in X_1 \times \ldots \times X_6$ which contains connectable trope numbers only we search the database in order to decide whether its D_6-orbit is the trope structure of a tone row.

2.3 The trope structure of a tone row

In order to display the trope $\{H_1, H_2\}$ consisting of the two hexachords H_1 and H_2 graphically, we draw the six vertices of the circular representation of $\mathbb{Z}_{12}$ belonging to H_1 in one color (say black) and the remaining six vertices in another color (say white). The number of its D_{12}-orbit is written in the center of the circular representation.

When displaying the trope structure of a tone row f graphically, we draw the six tropes τ_i, $i \in \{1, \ldots, 6\}$, defined by f, in a circle. The line between the τ_1 and τ_6 indicates, that the tropes corresponding to the remaining six shifts of f can be obtained by exchanging the colors in the first six diagrams.

2.4 Number of tropes in an n-scale

It is possible to enumerate orbits of tropes in an n-scale with respect to certain group actions (cf. Section 3.5.3 of [Fripertinger and Lackner(2015)]). Choose the action of a suitable group from dihedral group D_n, cyclic group C_n, linear affine group $\mathrm{Aff}_1(\mathbb{Z}_n)$, identity group which contains only the identity id on $\mathbb{Z}_n$, or an arbitrary group, which can be determined by a set of generators.

2.5 List of all tropes in an n-scale

For small values of n it is possible to compute a complete list of all tropes in $\mathbb{Z}_n$. Choose the action of a suitable group from dihedral group D_n, cyclic group C_n, linear affine group $\mathrm{Aff}_1(\mathbb{Z}_n)$, and decide whether the tropes should also be graphically displayed or not.

3 General orbits of tone rows

Instead of $D_{12} \times D_{12}$ we study the action of other groups on the set of all tone rows. Among these there are the following five groups which contain $D_{12} \times D_{12}$ as a proper subgroup.

- $\mathrm{Aff}_1(\mathbb{Z}_{12}) \times D_{12}$ which contains the quart-circle Q.
- $D_{12} \times \mathrm{Aff}_1(\mathbb{Z}_{12})$ which contains the 5-step F.
- $\mathrm{Aff}_1(\mathbb{Z}_{12}) \times \mathrm{Aff}_1(\mathbb{Z}_{12})$ which contains the 5-step F and the quart-circle Q.
- $\mathfrak{A}_{12}$ which contains the 5-step F, the quart-circle Q, and the exchange of parameters P.
- $\mathfrak{D}_{12}$ which contains the exchange of parameters P.

Furthermore also the following group actions on $\mathcal{R}$ can be studied.

- $C_{12} \times \langle R \rangle$ which is generated by $(T, 1)$ and $(1, R)$.
- $D_{12} \times \langle R \rangle$ which is generated by $(T, 1)$, $(I, 1)$, and $(1, R)$. These orbits correspond to Schönberg's notion of equivalent tone rows.
- $C_{12} \times C_{12}$ which is generated by $(T, 1)$ and $(1, R)$.
- $\langle I \rangle \times \langle 1 \rangle$ which is generated by $(I, 1)$.

- $\langle Q \rangle \times \langle 1 \rangle$ which is generated by $(Q, 1)$.
- $\langle 1 \rangle \times \langle R \rangle$ which is generated by $(1, R)$.
- $\langle 1 \rangle \times \langle F \rangle$ which is generated by $(1, F)$.
- $C_{12} \times \langle 1 \rangle$ which is generated by $(T, 1)$.
- $D_{12} \times \langle 1 \rangle$ which is generated by $(T, 1)$ and $(I, 1)$.
- $\mathrm{Aff}_1(\mathbb{Z}_{12}) \times \langle 1 \rangle$ which is generated by $(T, 1)$, $(I, 1)$ and $(Q, 1)$.
- $\langle 1 \rangle \times C_{12}$ which is generated by $(1, S)$.
- $\langle 1 \rangle \times D_{12}$ which is generated by $(1, S)$ and $(1, R)$.
- $\langle 1 \rangle \times \mathrm{Aff}_1(\mathbb{Z}_{12})$ which is generated by $(1, S)$, $(1, R)$ and $(1, F)$.
- $\langle P \rangle$ which is generated by the parameter exchange P.

It is possible to compute the orbit, the normal form or the stabilizer of a prescribed tone row which must be input in numeric form. For more details see Sections 3.1–3.3 of [Fripertinger and Lackner(2015)].

3.1 Decomposition into $D_{12} \times \langle R \rangle$-orbits (Schönberg-situation)

Since each $D_{12} \times D_{12}$-orbit of tone rows is a union of $D_{12} \times \langle R \rangle$-orbits it is possible to determine the decomposition of the $D_{12} \times D_{12}$-orbit of f into these $D_{12} \times \langle R \rangle$-orbits. Then the normal form of each $D_{12} \times \langle R \rangle$-orbit is tested whether it is an all-interval row, a combinatorial row, or a derived row. For more details see Section 3.7 of [Fripertinger and Lackner(2015)].

3.2 The row matrix of a tone row

In order to compute the row matrix of a tone row f, the tone row must be input in numeric form. The rows and columns of this matrix contain all elements of the $D_{12} \times \langle R \rangle$-orbit of f (Schönberg situation). E.g. the normal form of this orbit stands in the first row of this matrix.

3.3 Operation on a tone row

An arbitrary element of $\mathfrak{A}_{12}$ is of the form $(T^i I^j Q^k, S^m R^n F^r) \circ P^s$ with $i, m \in \{0, 1, \ldots, 11\}$, $j, k, n, r, s \in \{0, 1\}$. In order to compute

$$(T^i I^j Q^k, S^m R^n F^r) \circ P^s * f = T^i \circ I^j \circ Q^k \circ (P^s \circ f) \circ F^r \circ R^n \circ S^{-m}$$

of a tone row, input f in numeric form and the values of the exponents i, j, k, m, n, r, s. According to the definition of this group action the sequence of $S^m R^n F^r$ acting on the domain of f must be reversed and the exponent m must be replaced by $-m$. Since $r, n \in \{0, 1\}$ the minus can be omitted in front of r or n.

3.4 Number of tone rows in an n-scale

It is possible to enumerate orbits of tone rows in an n-scale with respect to certain group actions. Both for the domain and the range of tone rows choose the action of a suitable group from dihedral group D_n, cyclic group C_n, linear affine group $\mathrm{Aff}_1(\mathbb{Z}_n)$, identity group which contains only the identity id on $\mathbb{Z}_n$, or an arbitrary group, which can be determined by a set of generators. As a matter of fact the degrees of both permutation groups must be the same!

4 Bigger orbits

We collect information and describe relations between the $D_{12} \times D_{12}$-, $\mathrm{Aff}_1(\mathbb{Z}_{12}) \times D_{12}$-, $D_{12} \times \mathrm{Aff}_1(\mathbb{Z}_{12})$-, $\mathrm{Aff}_1(\mathbb{Z}_{12}) \times \mathrm{Aff}_1(\mathbb{Z}_{12})$-, $\mathfrak{D}_{12}$-, and $\mathfrak{A}_{12}$-orbits of tone rows. For more details see Section 4 of [Fripertinger and Lackner(20

4.1 Information on bigger orbits

Here we describe how the orbits of tone rows under bigger groups decompose into $D_{12} \times D_{12}$-orbits.

First the acting group G must be chosen. The tone rows we are interested in must be input by the numbers of their $D_{12} \times D_{12}$-orbits from $\{1, \ldots, 836\,017\}$. It is also possible to search for all tone rows with prescribed stabilizer type (with respect to the chosen group action). The stabilizer types are described in the online document.

These bigger groups collect 1, 2, 4, or 8 different $D_{12} \times D_{12}$-orbits to one bigger orbit. It is also possible to restrict the search to tone rows which belong to bigger orbits collecting a prescribed number of smaller orbits. The different input blocks are connected with `AND`.

For each search result it will be possible to open pop-up windows showing

- the decomposition of the G-orbit into $D_{12} \times \langle R \rangle$-orbits,
- musical information on the G-orbit
- all elements of the G-orbit,
- the stabilizer of the normal form of the G-orbit,
- the chromatic circular representation of all $D_{12} \times D_{12}$-orbits contained in the G-orbit.

4.2 Simultaneous information on bigger orbits

This program displays simultaneously information on the $D_{12} \times D_{12}$-, $\mathrm{Aff}_1(\mathbb{Z}_{12}) \times D_{12}$-, $D_{12} \times \mathrm{Aff}_1(\mathbb{Z}_{12})$-, $\mathrm{Aff}_1(\mathbb{Z}_{12}) \times \mathrm{Aff}_1(\mathbb{Z}_{12})$-, $\mathfrak{D}_{12}$-, and $\mathfrak{A}_{12}$-orbits of tone rows. It allows to search for $D_{12} \times D_{12}$-orbits of tone rows given by their number from $\{1, \ldots, 836\,017\}$, or to prescribe or exclude the stabilizer types with respect to the five group actions above.

If different $D_{12} \times D_{12}$-orbits are collected under the action of a bigger group to a bigger orbit, then it is possible to determine permuting elements between the $D_{12} \times D_{12}$-normal forms of these orbits by opening a pop-up window.

If the $D_{12} \times D_{12}$-orbit of a tone row f coincides with its orbit under a bigger group, then for all additional generators h of the bigger group it is possible to determine elements $g \in D_{12} \times D_{12}$ so that $g * f = h * f$ by opening a pop-up window.

The output consists of a big table. The first column contains the number of the $D_{12} \times D_{12}$-orbit, the second column the stabilizer type of this orbit. The third column displays the $\mathrm{Aff}_1(\mathbb{Z}_{12}) \times D_{12}$-orbit. Usually it consists of two $D_{12} \times D_{12}$-orbits, the one from the first column and an additional orbit whose number is written in the third column. If this number is smaller than the number in the first column, then this new orbit contains the normal form of the $\mathrm{Aff}_1(\mathbb{Z}_{12}) \times D_{12}$-orbit. In this case its number will be printed in bold face. If the first column contains the normal form of the $\mathrm{Aff}_1(\mathbb{Z}_{12}) \times D_{12}$-orbit, then in the fourth column the stabilizer type of this orbit is indicated. The next two columns describe the $D_{12} \times \mathrm{Aff}_1(\mathbb{Z}_{12})$-orbit in a similar way. The following two columns describe the $\mathrm{Aff}_1(\mathbb{Z}_{12}) \times \mathrm{Aff}_1(\mathbb{Z}_{12})$-orbit which consists of the $D_{12} \times D_{12}$-orbit from the first column and up to three further $D_{12} \times D_{12}$-orbits. The following four columns describe the $\mathfrak{A}_{12}$- and $\mathfrak{D}_{12}$-orbits in a similar way.

4.3 Find a permuting element

Input a vector containing at least two tone rows. For each of the input tone rows with exception of the first one the program tries to find an element of the chosen acting group which applied to the first tone row yields the corresponding tone row.

4.4 Bigger orbits with high symmetry

For a tone row f given in numeric form it is checked whether the quart circle, the five-step, or the exchange of parameters of f already belongs to the $D_{12} \times D_{12}$-orbit of f or not.

5 Chords

Let n be a positive integer. For $0 \leq k \leq n$ a k-chord is a k-set of the the n-scale $\mathbb{Z}_n$. If G is a group acting on $\mathbb{Z}_n$, then G acts in a natural way on the set of all k-chords of $\mathbb{Z}_n$.

5.1 Number of k-chords in an n-scale

It is possible to enumerate orbits of k-chords in an n-scale with respect to certain group actions. Choose the action of a suitable group from dihedral group D_n, cyclic group C_n, linear affine group $\mathrm{Aff}_1(\mathbb{Z}_n)$, identity group which contains only the identity id on $\mathbb{Z}_n$, or an arbitrary group, which can be determined by a set of generators.

The result is a polynomial in x with integer coefficients. The coefficient of x^k is the number of orbits of k-chords, $0 \leq k \leq n$.

5.2 List of all k-chords in an n-scale

For small values of n it is possible to compute a complete list of all k-chords in $\mathbb{Z}_n$, $0 \leq k \leq n$. Choose the action of a suitable group from dihedral group D_n, cyclic group C_n, linear affine group $\mathrm{Aff}_1(\mathbb{Z}_n)$, and decide whether the k-chords should also be graphically displayed or not.

5.3 Normal form of a k-chord

Please input n, the number of tones in the scale, k, the number of tones in the chord, the chord consisting of k pitch classes, and the acting group. For the labeling of the pitch classes use either the symbols $1, \ldots, n$, or the symbols $0, \ldots, n-1$. A k-chord is a comma-separated list of k distinct pitch classes.

6 Musical information

We were collecting data on tone rows appearing in works of various composers. Therefore it is possible to check

- whether the tone rows appearing in the search results were used by other composers and were already collected to our database.
- whether there are properties of tone rows common to different tone rows of one composer.
- whether different composers used similar orbits of tone rows, where similarity means, that the tone rows belong to the same orbit. In order to study different degrees of similarity we consider the $D_{12} \times D_{12}$-orbits as similarity of highest degree, and the $\mathfrak{A}_{12}$-orbits, which are the biggest orbits, as the weakest degree of similarity.

The following facts are stored in our database:

- The individual tone row, the number of its $D_{12} \times D_{12}$-orbit, and its position in this orbit.
- The name of the composer, and sometimes also a web-link to further information on the composer.
- The title of the composition, and sometimes also a web-link to further information on the composition.
- The date when the piece was composed.
- Some further information if available.

As a matter of fact, at the moment we have more than 1200 entries of musical information in our database. Of course this is not enough for doing some statistical analysis or to suggest trends in the usage of certain types of tone rows. However, we try to collect further tone rows and data. Therefore it is possible to input new musical information by the different users of the database.

There are more than 500 entries with tone rows by J.M. Hauer. All tone rows from the Second Viennese School and a selection of compositions until today are input.

6.1 Search the database for musical information

It is possible to search

- for a tone row given by the number of its orbit from $\{1, \ldots, 836\,017\}$.
- for a tone row input in numeric form. In this case a composition is found when exactly this tone row (and not another member of its $D_{12} \times D_{12}$-orbit) is stored in the database.
- for a composer by his/her surname and first name.
- for the title of a composition. If several words are input, then all words must occur in the title of the composition.
- for the compositions from a particular year.
- for entries in the field "further information". If several words are input, then all words must occur in the field "further information".

Different search criteria are connected with `AND`.

It is not necessary to use capital letters. Special symbols like "ä", "ö", "ü", or "ß" can be written as "ae", "oe", "ue" or "ss". An arbitrary single letter can be replaced by ".", several arbitrary letters by "`.*`" or "`.+`". It is not necessary to write the complete word. E.g. searching for composers with `Schoe` yields results for Schönberg, just searching for `Sch` yields results for Schönberg but also other composers like Schnittke. If the input part of the word does not necessarily stand at the beginning of the word use `.*` in front of it. E.g. searching for composers with `.*berg` gives results for Schönberg and Berg, whereas searching for composers with `.+berg` does not show Berg.

If no search criteria are inserted, then all musical information will be displayed.

6.2 Input musical information into the database

In order to input new data, as many fields as possible should be filled. It is necessary to input a tone row in numeric form which in general should not be given in a normal form, but in this form in which it appears in the composition. For the different input alphabets the translation

	c	c♯	d	d♯	e	f	f♯	g	g♯	a	a♯	b
A_1	1	2	3	4	5	6	7	8	9	10	11	12
A_2	0	1	2	3	4	5	6	7	8	9	10	11
A_3	0	1	2	3	4	5	6	7	8	9	A	B

should be used. Furthermore the name of the composer, his/her first name, the title of the composition, and the date when the piece was written must be input. Some additional information can be input, e.g. concerning the scoring of the piece, the name of the tone row, or the part where the tone row occurs. Optionally a link to the composer and/or to the composition can be added. If all the data are input correctly, then the software will determine the number of the $D_{12} \times D_{12}$-orbit of the tone row, and its position in the orbit. The new entry will not immediately be open to the public. First it must be checked by the second author. Obviously, often it is necessary to analyze a composition thoroughly until the tone row it is based on can be detected. Therefore it would be helpful to obtain a scan of the composition per e-mail showing the occurrence of the tone row.

References

[Fripertinger and Lackner(2015)] Fripertinger, Harald, and Peter Lackner. 2015. "Tone rows and tropes." To appear in the Journal of Mathematics and Music.